Math 1552

Sections 10.1: Sequences

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today's Learning Goals

- Use proper notation to denote a sequence.
- Understand how to find lower and upper bounds for sequences.
- Determine if a sequence is monotonic.
- Find limits of sequences when possible.

Sequences

A *sequence* is a *function* from the set of positive integers to the set of real numbers.

$$\{a_n\} = \{f(n)\} = a_1, a_2, \ldots, a_k, \ldots$$

a_n is called the n^{th} term

OR

$$\{a_n\}_{n\geq 1} = \{a_1, a_2, a_3, \ldots\}$$

$$\{f(n)\}_{n=0}^{\infty} = \{f(0), f(1), f(2), \ldots\}$$

The values of *n* are all positive integers, unless otherwise specified, e.g., starting from n=0 in the third form above.

Example:

Find an expression for the general term of the sequence below:

$$-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\frac{5}{6},\ldots$$

A) $a_n = \dfrac{(-1)^n n}{n+1}$

B) $a_n = \dfrac{(-1)^{n+1} n}{n+1}$

C) $a_n = \dfrac{(-1)^n (n+1)}{n+2}$

D) $a_n = \dfrac{(-1)^{n+1} (n+1)}{n+2}$

LUB and GLB

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (*l.u.b.*) – *cf. the supremum*.

LUB and GLB

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (*l.u.b.*) – *cf. the supremum*.
- A *lower bound* of a set S is a number *m* that is less than or equal to each element in S.
- The largest possible lower bound is called the *greatest lower bound* (*g.l.b.) – cf. the infimum.*

Example:

Find the l.u.b. and g.l.b. of the sequence: $\left\{ \dfrac{n+1}{n} \right\}$

A. l.u.b.=1, g.l.b.=0
B. l.u.b.=2, g.l.b.=0
C. l.u.b.=2, g.l.b.=1
D. No l.u.b., g.l.b.=0

Monotone Sequences

A sequence is called ***monotonic*** if any one of the following statements holds:

$(i)\ a_n < a_{n+1}$ for all n (strictly increasing) *(cf. non-decreasing)*

$(ii)\ a_n \le a_{n+1}$ for all n (monotonically increasing) *(cf. non-decreasing)*

$(iii)\ a_n > a_{n+1}$ for all n (strictly decreasing) *(cf. non-increasing)*

$(iv)\ a_n \ge a_{n+1}$ for all n (monotonically decreasing) *(cf. non-increasing)*

Limit of a Sequence

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} a_n = L$,

then L is the ***limit*** of this sequence.

If the sequence has a finite limit *L*, then the sequence is said to ***converge*** to L.

Otherwise, the sequence is said to ***diverge***.

Convergence Theorem

If a sequence $\{a_n\}_{n\geq 0}$ is ***monotonic*** and ***bounded***,
then it converges (*to some finite limit L*).

If the sequence is *increasing*, then *L*=l.u.b.

If the sequence is *decreasing*, then *L*=g.l.b.

Equivalent statement:

An unbounded sequence diverges.

Example A: Determine whether the sequence converges. If so, find the limit. $\left\{\dfrac{n^2}{n+1}\right\}_{n\geq 1}$

Example B: Determine whether the sequence converges. If so, find the limit. $\{(-3)^n\}_{n\geq 1}$

Example C:

Determine whether the sequence converges. If so, find the limit.

$$\left\{ \frac{(-1)^n}{2^n} \right\}_{n \geq 1}$$

Example D:

Determine whether the sequence converges.

If so, find the limit. $\left\{ \dfrac{2^n}{n!} \right\}_{n \geq 1}$

Example E:

Determine whether the sequence converges. If so, find the limit. $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n\geq 1}$

Example: Find the limit of the following sequence, if it exists: $\left\{\dfrac{2n+1}{1-3n}\right\}$ *(Justify your answer carefully.)*

A. 0
B. -2/3
C. 2/3
D. Diverges

Some Common Limits (memorize)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

7) $\lim_{n\to\infty} n^{1/n} = 1$

8) If p is a positive integer, then:

$$\lim_{n\to\infty} \frac{a_p n^p + \cdots + a_1 n + a_0}{b_p n^p + \cdots + b_1 n + b_0} = \frac{a_p}{b_p}$$

(Do you see why?)

An interesting example

Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

An interesting example

Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

Consider using a Riemann sum to approximate the sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.

Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

An interesting example

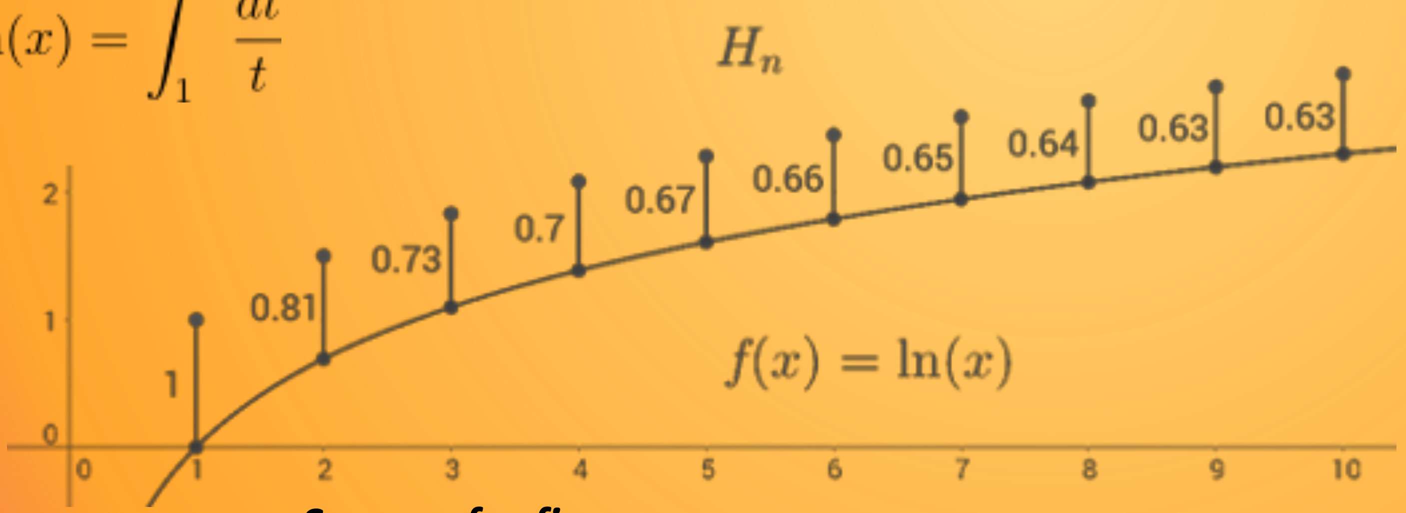
Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

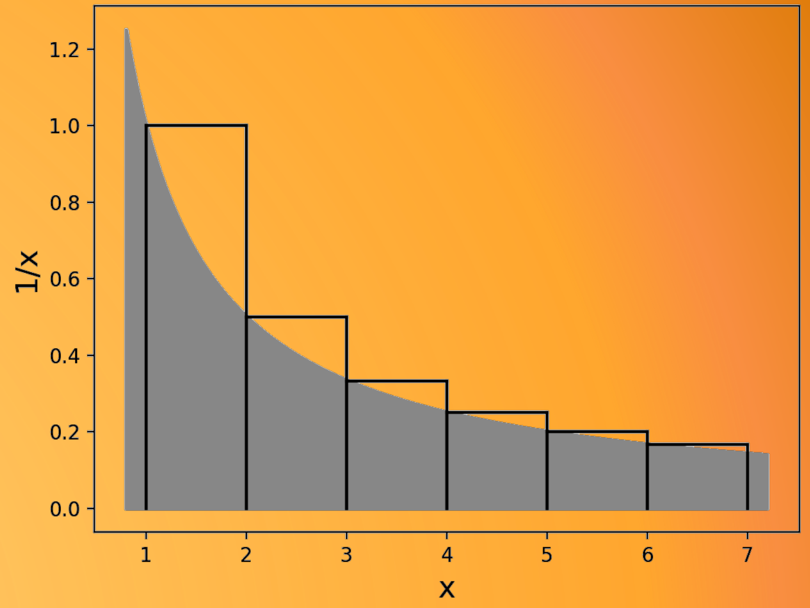
Consider using a Riemann sum to approximate the sums

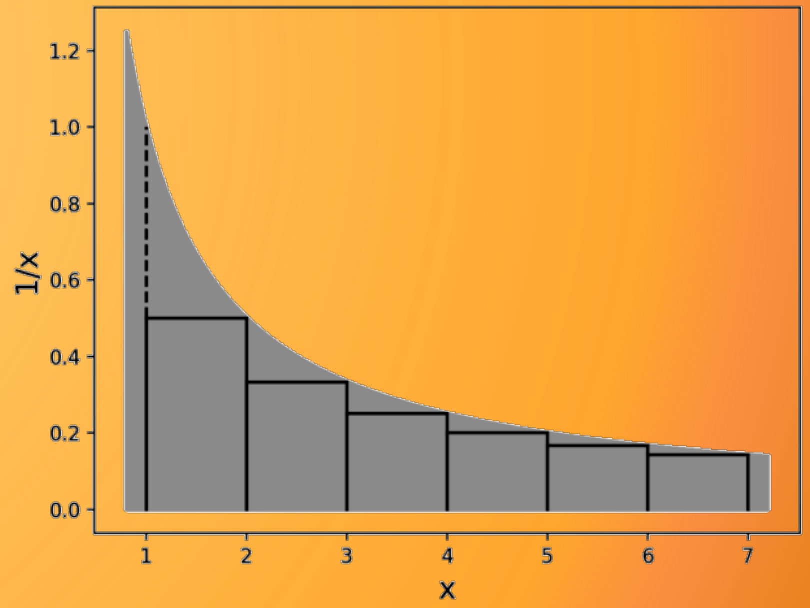
$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.

Recall: $\ln(x) = \int_1^x \frac{dt}{t}$







Sources for figures:
https://www.cantorsparadise.com/the-euler-mascheroni-constant-4bd34203aa01
https://brilliant.org/wiki/euler-mascheroni-constant/

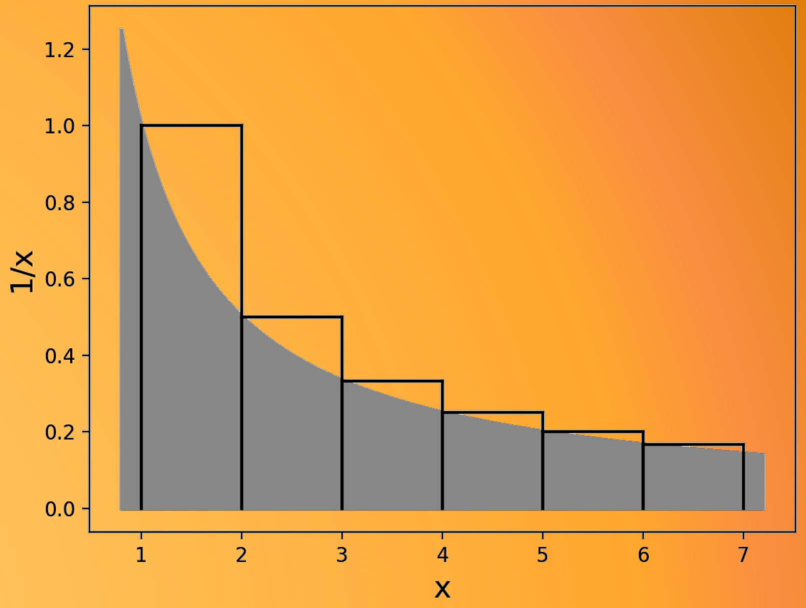
<u>*An interesting example*</u>

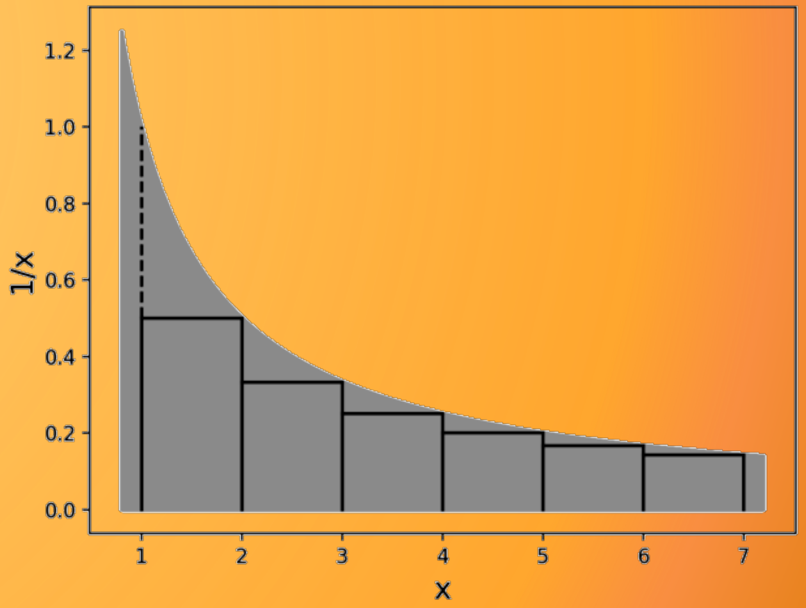
Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

Consider using a Riemann sum to approximate the sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.





Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

Define: $T_n = H_n - \ln n$

We can show using <u>elementary methods</u> that

$$0 < \frac{1}{n} < T_n < 1, \text{ for all } n \geq 1$$

AND

$$T_{n+1} < T_n, \text{ for all } n \geq 1$$

$$\implies \gamma = \lim_{n \to \infty} T_n$$ ***EXISTS!***

(by monotonicity and boundedness)

An interesting example

Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

Consider using a Riemann sum to approximate the sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.

Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

Define: $T_n = H_n - \ln n$

We can show using <u>elementary methods</u> that

$$0 < \frac{1}{n} < T_n < 1, \text{ for all } n \geq 1$$

AND

$$T_{n+1} < T_n, \text{ for all } n \geq 1$$

$$\implies \gamma = \lim_{n\to\infty} T_n$$

EXISTS!
(by monotonicity and boundedness)

This constant is called *Euler-Mascheroni's gamma* (or the *Euler gamma* constant for short):

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right]$$
$$= \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx$$
$$\approx 0.5772156649015328606065120$$

An interesting example

Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

Consider using a Riemann sum to approximate the sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.

Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

Define: $T_n = H_n - \ln n$

We can show using <u>elementary methods</u> that

$$0 < \frac{1}{n} < T_n < 1, \text{ for all } n \geq 1$$

AND

$$T_{n+1} < T_n, \text{ for all } n \geq 1$$

$$\Longrightarrow \gamma = \lim_{n\to\infty} T_n$$

EXISTS!
(by monotonicity and boundedness)

This constant is called *Euler-Mascheroni's gamma* (or the *Euler gamma* constant for short):

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \\ &= \int_1^{\infty}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx \\ &\approx 0.5772156649015328606065120\end{aligned}$$

The *harmonic series* is an example of a p-series (with p=1) that diverges.
Now you can see why! ☺

Challenge problem on limits of sequences I:

Suppose that $a_n = \begin{cases} 1 + \frac{1}{a_{n-1}}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\lim_{n\to\infty} a_n$?

Challenge problem on limits of sequences II:

Suppose that $b_n = \begin{cases} b_{n-1} + 2b_{n-2}, & \text{if } n \geq 2; \\ 1, & \text{if } n = 1; \\ 2, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\lim_{n \to \infty} b_n$?

Bonus problems on limits I:

Evaluate the following limit: $$\lim_{n\to\infty}\left(1-\frac{x}{n}+\frac{x^2}{n^2}\right)^n$$

Bonus problems on limits II:

Evaluate the following limit: $$\lim_{n\to\infty}\left(1+\frac{x^2}{n^2}\right)^{n^2}$$

Bonus problems on limits III (extra):

Show that $$\lim_{\alpha\to 0^+}\left(\frac{1-e^{-\alpha v}}{\alpha}\right)^x = v^x, \alpha > 0, v > 0$$

Math 1552

Sections 10.1:
Review of Sequences

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review Question: Which of the following sequences converge?

(A) $\left\{\dfrac{2n+1}{1-3n}\right\}$

(B) $\left\{(-1)^n\right\}$

(C) $\left\{\dfrac{2^n}{n!}\right\}$

(D) $\left\{\left(1+\dfrac{4}{n}\right)^n\right\}$

Math 1552

Sections 10.2: Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Learning Goals

- Understand what is meant by an infinite series
- Understand the general rule of when an infinite series converges
- Identify geometric series and find their sums
- Identify telescoping series and find their sums
- Determine convergence or divergence with the nth term test

Recall: Limit of a Sequence

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n = L$,

then L is the ***limit*** of this sequence.

If the sequence has a finite limit *L*, then the sequence is said to ***converge*** to L.

Otherwise, the sequence ***diverges***.

Review of Sigma Notation

Recall from the sections on Riemann sums that

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

$$\sum_{k=1}^{n} 1 = n, \text{so} \sum_{k=0}^{n} 1 = n + 1$$

$$\sum_{k=m}^{n} \left(a_k + b_k\right) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

$$\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$$

(Linearity)

$$\sum_{k=0}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=0}^{n} a_k$$

(Linearity)

Infinite Series

An *infinite series* is a *sum* of infinitely many terms:

$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

Infinite Series

An *infinite series* is a *sum* of infinitely many terms:

$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

The series *converges* if the sequence of partial sums converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = L$$

The series *diverges* otherwise.

Which of these series do you think converges?

(That is, à priori – we will cover precise criteria for each case in the next slides.)

(A) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}$

(C) $\displaystyle\sum_{\mathrm{n}=0}^{\infty}\left(\frac{2}{3}\right)^{n}$

(D) None of these

The Harmonic Series

The ***Harmonic Series*** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ***diverges*****!**

(Recall that we saw a proof of this fact in the Week 5 slides!)

Telescoping Series

- A telescoping series has the form:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}$$

- These series ***converge***.
- To find the sum, use *partial fractions*.

An Example:

Evaluate the following sum: $$S = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2^k (3k+1)}{(k+1)(k+3)}$$

Geometric Series

- A geometric series has the form:

$$\sum_{n=0}^{\infty} r^n$$

- It ***converges* when $|r|<1$** and ***diverges*** otherwise.
- If $|r|<1$, the sum is:

$$\frac{1}{1-r}.$$

Example 1.1:

Sum the series: $$\sum_{n=2}^{\infty} \frac{5^{n-1} + 3 \cdot 2^{3n}}{9^n}$$

Example 1.2:

Use series to write the decimal 1.42424242… as a *rational number*.

Divergence (n^{th} term) Test

Given $\sum_{n=0}^{\infty} a_n$, first find $\lim_{n\to\infty} a_n$.

If $\lim_{n\to\infty} a_n \neq 0$, then the series **DIVERGES**!

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

<u>Important:</u> *n^{th} term test only tests for divergence!!*

- If the limit of the terms is equal to 0, you do not have enough information!
- For instance:
  - The harmonic series, the terms go to 0 but the series diverges!
  - Telescoping series, the terms go to 0 and these series converge!
- So… in order to converge, we need the limit to go to zero, but it is NOT a sufficient condition to determine convergence!

Example A:

Does the series diverge by the n^{th} term test?

$$\sum_{k=1}^{\infty}\left(1+\frac{3}{k}\right)^{k}$$

Example B:

Does the series diverge by the n^{th} term test?

$$\sum_{k=2}^{\infty} \frac{3k}{5k-7}$$

Example C:

Does the series diverge by the n^{th} term test?

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+6}}$$

Which statement is always true?

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then:}$$

A. The series converges.
B. The sequence converges.
C. The sequence of partial sums converges.
D. The series diverges.

Some Convergence Theorems

(1) If $\sum a_n$ and $\sum b_n$ both converge, then $\sum (a_n \pm b_n)$ also converges.

(2) If $\sum a_n$ converges, then $\sum c a_n$ also converges for any $c \in \Re$.

(3) If $\sum_{n=j}^{\infty} a_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.

Math 1552

Sections 10.3, 10.4 and 10.5 : Convergence Tests for Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Learning Goals

- Learn how to apply the integral, comparison, limit comparison, ratio and root series to determine whether an infinite series converges or diverges
- Learn when to apply which test
- Summarize the results into a formal mathematical justification

Quick review…

- The harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ DIVERGES.

Quick review…

- The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \begin{array}{l} \textit{converges} \text{ to } \dfrac{1}{1-r} \text{ when } |r|<1 \\ \textit{diverges} \text{ when } |\mathrm{r}| \geq 1 \end{array}$$

Divergence (n^{th} term) Test

Given $\sum_{k=0}^{\infty} a_k$, first find $\lim_{n\to\infty} a_n$.

If $\lim_{n\to\infty} a_n \neq 0$, then the series **DIVERGES**!

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

Integral Test

Let f be a continuous, positive, and decreasing function. Then:

$$\sum_{k=1}^{\infty} f(k) \textit{ converges} \text{ if and only if } \int_1^{\infty} f(x)dx \text{ converges,}$$

$$\text{and } \textit{diverges} \text{ if and only if } \int_1^{N} f(x)dx \to \infty \text{ as } N \to \infty.$$

Example I:

Use the integral test to determine whether the series converges: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

Example II:

When does a p-series converge? $$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ (p - series)}$$

Series we know:

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \textit{converges} \text{ to } \frac{1}{1-r} \text{ when } |r|<1$$

$$\textit{diverges} \text{ when } |\,\mathrm{r}\,| \geq 1$$

- A p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \textit{converges when } p>1$$

$$\textit{diverges} \text{ when } p \leq 1$$

Some Convergence Theorems

(1) If $\sum a_k$ and $\sum b_k$ both converge, then $\sum (a_k \pm b_k)$ also converges.

(2) If $\sum a_k$ converges, then $\sum c a_k$ also converges for any $c \in \Re$.

(3) If $\sum_{k=j}^{\infty} a_k$ converges, so does $\sum_{k=0}^{\infty} a_k$.